



# The $\kappa$ -nullity of Riemannian manifolds and their splitting tensors

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## Abstract

We consider Riemannian  $n$ -manifolds  $M$  with nontrivial  $\kappa$ -nullity “distribution” of the curvature tensor  $R$ , namely, the variable rank distribution of tangent subspaces to  $M$  where  $R$  coincides with the curvature tensor of a space of constant curvature  $\kappa$  ( $\kappa \in \mathbb{R}$ ) is nontrivial. We obtain classification theorems under different additional assumptions, in terms of low nullity/conullity, controlled scalar curvature or existence of quotients of finite volume. We prove new results, but also revisit previous ones.

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## 1 Introduction

Several important classes of Riemannian manifolds  $M$  are defined by imposing a certain condition on its Riemann curvature tensor  $R$ , such as spaces of constant curvature, Einstein manifolds, locally symmetric spaces, etc. In a somehow different sense, it is a stimulating problem to define a class of Riemannian manifolds by imposing a certain *form* on their curvature tensors. More specifically, let  $T$  be an algebraic curvature tensor. A Riemannian manifold  $M$  is said to be *modelled* on  $T$  if its curvature tensor is, at each point, orthogonally equivalent to  $T$ . Here the size of the orbit of  $T$  under the action of the orthogonal group plays a certain role; for instance, curvature tensors of spaces of constant curvature are fixed points

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of that action, and in this case a manifold modelled on  $T$  will obviously also have constant curvature. On the other hand, if  $T$  is only required to be the curvature tensor of a homogeneous Riemannian manifold  $\bar{M}$ , there are continuous families of examples of complete irreducible Riemannian manifolds  $M$  modelled on  $T$  which are not locally isometric to  $\bar{M}$  (see e.g. [24] for examples and a discussion of related results, which originate from a question of Gromov).

If a Riemannian manifold  $M$  is modelled on an algebraic curvature tensor  $T$ , then clearly it is also *curvature homogeneous*, in the sense that the curvature tensors at any two of its points are orthogonally equivalent. The totality of curvature homogeneous manifolds (for varying  $T$ ) obviously include locally homogeneous spaces, but contains strictly more manifolds. The first examples were constructed by Takagi [41] and Sekigawa [36], in response to a question by Singer (these were later generalized, see [3] for the full range of generalizations).

In a different vein, a Riemannian manifold is called *semi-symmetric* if its curvature tensor is, at each point, orthogonally equivalent to the curvature tensor of a symmetric space; the symmetric space may depend on the point (in particular a curvature homogeneous semi-symmetric space is a Riemannian manifold modelled on the curvature tensor of a fixed symmetric space). In 1968, Nomizu conjectured that every complete irreducible semi-symmetric space of dimension greater than or equal to three would be locally symmetric. His conjecture was refuted by Takagi [40] and Sekigawa [35], who constructed counterexamples (see [3] for further developments). The complete classification of semi-symmetric spaces is the work of Z. I. Szabó [39]. On the other hand, Florit and Ziller have shown that the Nomizu conjecture holds for manifolds of finite volume [19].

It is remarkable what all of the examples above (and others) have in common, namely, their curvature tensor has a large nullity. This leads us to the class of Riemannian manifolds that we consider herein; loosely speaking, we say a Riemannian manifold has *non-trivial  $\kappa$ -nullity*, where  $\kappa \in \mathbb{R}$ , if the variable rank tangent distribution where its curvature tensor behaves like that of a space of constant curvature  $\kappa$  is non-trivial (as an extrinsic counterpart to the above examples, recall that, owing to the Beez-Killing theorem, a locally deformable hypersurface in a space form of curvature  $\kappa$ , without isotropic points, has precisely two nonzero principal curvatures at each point, and hence has a  $\kappa$ -nullity distribution of codimension 2).

The idea of nullity was introduced in case  $\kappa = 0$  by Chern and Kuiper in [11], and for general  $\kappa$  by Otsuki [30], and later reformulated and studied by different authors (see e.g. [20, 25] and, for more recent work, [14, 15, 19] and the references therein). Each sign of  $\kappa$  (positive, negative or zero) yields results of a different flavor. In this paper we consider the three cases, and note that the concept of nullity has connections with diverse areas such as Sasakian manifolds, solvmanifolds, and non-holonomic geometry. Our main tool is the so called *splitting tensor* (cf. section 2). We prove new results, but we also aim to extend, unify and simplify existing results in the literature.

More precisely, let  $M$  be a connected Riemannian manifold, and consider the curvature tensor  $R$  of its Levi-Civita connection  $\nabla$  with the sign convention

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

for vector fields  $X, Y, Z \in \Gamma(TM)$ . For  $\kappa \in \mathbb{R}$ , the  $\kappa$ -nullity distribution of  $M$  is the variable rank distribution  $\mathcal{N}_\kappa$  on  $M$  defined for each  $p \in M$  by

$$\mathcal{N}_\kappa|_p = \{z \in T_p M : R_p(x, y)z = -\kappa(\langle x, z \rangle_p y - \langle y, z \rangle_p x) \text{ for all } x, y \in T_p M\}.$$

The number  $v_\kappa(p) := \dim \mathcal{N}_\kappa|_p$  is called the *index of  $\kappa$ -nullity at  $p$* .

In case  $\kappa = 0$  we obtain trivial examples of manifolds with positive  $v_0$  simply by taking a Riemannian product with an Euclidean space, but similar product examples do not occur

if  $\kappa \neq 0$ . It is easily seen that  $v_\kappa(p)$  is nonzero for at most one value of  $\kappa$ . For general  $M$ ,  $v_\kappa$  is nonnecessarily constant if nonzero, but it is an upper semicontinuous function, so there is an open and dense set of  $M$  where  $v_\kappa$  is locally constant, and there is an open subset  $\Omega$  of  $M$  where  $v_\kappa$  attains its minimum value. It is known that  $\mathcal{N}_\kappa$  is an autoparallel distribution on any open set where  $v_\kappa$  is locally constant and, in case  $M$  is a complete Riemannian manifold, its leaves in  $\Omega$  are *complete* totally geodesic submanifolds of constant curvature  $\kappa$  [25].

We call the orthogonal complement of  $\mathcal{N}_\kappa$  the  $\kappa$ -*conullity distribution* of  $M$ , and its dimension at a point  $p \in M$  the *index of  $\kappa$ -conullity* at  $p$ , or simply, the  $\kappa$ -conullity at  $p$ . For obvious reasons, the minimal nonzero value of the  $\kappa$ -conullity is 2. Riemannian manifolds with 0-conullity at most 2 have pointwise the curvature tensor of an isometric product of Euclidean space with a surface with constant curvature and hence are semi-symmetric. Conversely, a complete irreducible semi-symmetric space is either locally symmetric or has 0-conullity at most 2 in an open and dense subset [39].

In our study we apply a homothety and assume that  $\kappa$  is equal to  $+1$ ,  $-1$  or  $0$ . Generally speaking, the results below give characterizations/classifications of manifolds in terms of low conullity/nullity, controlled scalar curvature and/or existence of quotients of finite volume. Some terminology: in general we shall say an  $n$ -manifold has *minimal  $\kappa$ -nullity*  $d$  (resp. *maximal  $\kappa$ -conullity*  $n - d$ ) to mean that  $v_\kappa \geq d$  everywhere and the equality holds at some point.

## 1.1 Results with $\kappa = +1$

The following theorem gives a lot of rigidity in the case of constant  $(+1)$ -conullity 2 and constant scalar curvature. It should be compared with the examples constructed in [38] of certain inhomogeneous conformal deformations of left-invariant metrics on  $SU(2)$ ,  $\widetilde{SL(2, \mathbb{R})}$ , and  $Nil^3$ , which posses  $(+1)$ -conullity 2 and nonconstant scalar curvature.

**Theorem 1.1** *Let  $M$  be a simply-connected complete Riemannian  $n$ -manifold with constant  $(+1)$ -conullity equal to 2, and constant scalar curvature. Then  $M$  is a 3-dimensional Sasakian space form, that is, isometric to one of the Lie groups  $SU(2)$  (the Berger sphere),  $\widetilde{SL(2, \mathbb{R})}$  (the universal covering of the unit tangent bundle of the real hyperbolic space), or  $Nil^3$  (the Heisenberg group). In all cases, the  $(+1)$ -nullity distribution is orthogonal to the contact distribution.*

**Corollary 1.1** *A complete Riemannian manifold modelled on one of the left-invariant metrics listed on Table 1 is locally isometric to the corresponding model.*

Recall that a Sasakian space form is a Sasakian manifold of constant  $\varphi$ -sectional curvature, where the  $\varphi$ -sectional curvature plays the role accorded to the holomorphic sectional curvature in Kähler geometry. We refer to [4] for a discussion of Sasakian geometry. In particular the spaces in Theorem 1.1 have the structure of Lie groups and thus are homogeneous contact metric manifolds. A straightforward computation using [26] shows that the admissible metrics with 1-conullity 2 are given as follows. The groups  $SU(2)$ ,  $\widetilde{SL(2, \mathbb{R})}$ ,  $Nil^3$  are unimodular, so there is an orthonormal basis  $e_1, e_2, e_3$ , where  $e_1$  is tangent to the 1-nullity and

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2. \quad (1.1)$$

By changing  $e_1$  to  $-e_1$ , we may assume  $\lambda_1 > 0$ , and then the possibilities for left-invariant metrics are given in Table 1.

According to Perrone [31], there is an additional, non-unimodular Lie group structure on the Sasakian space form of  $\varphi$ -sectional curvature  $< -3$ , that is, the simply-connected solvable Lie group with Lie algebra

$$[e_1, e_2] = \alpha e_2 + 2\xi, \quad [e_1, \xi] = [e_2, \xi] = 0,$$

where  $\alpha \neq 0$ ,  $\xi$  is the characteristic vector field and spans  $\mathcal{N}_1$ , and  $e_1, e_2, \xi$  is orthonormal, is isometric (but not isomorphic) to  $\widetilde{SL}(2, \mathbb{R})$  if  $\alpha^2 = -2\theta$ .

Riemannian manifolds with non-trivial (+1)-nullity are also considered in [12, 29] under different assumptions. As an application of the results in [29], the authors obtain the classification of simply-connected complete Riemannian manifolds  $M$  with non-trivial (+1)-nullity with the condition that  $M$  is the total space of a Riemannian submersion whose fibers are the integral leaves of  $\mathcal{N}_1$ ; the cases of index of conullity 2 in [29, Table 3] correspond to having the base of the Riemannian submersion to be two-dimensional. In [12] is introduced a closely related class of Riemannian manifolds  $M$ , called *n-Sasakian*, which means that  $M$  is foliated by totally geodesic equidistant  $n$ -manifolds such that the leaves are contained in the (+1)-nullity of  $M$  (see [13, Definition 13]); some examples are provided, coming from circle quotients of certain focal sets of isoparametric submanifolds in spheres, and they in general have large (+1)-conullity.

## 1.2 Results with $\kappa = 0$

The 3-dimensional case of the following theorem is proved in [1, Thm. 3], and a simple proof in the case of arbitrary dimension is sketched in [19, Remark, p. 1324]. For the convenience of the reader, we provide an alternate, and as well simple, argument in subsection 3.2.

**Theorem 1.2** *Let  $M$  be a simply-connected complete Riemannian  $n$ -manifold with maximal 0-conullity 2. Assume the scalar curvature function  $\text{scal}$  is positive and bounded away from zero. Then  $M$  splits as the Riemannian product  $\mathbb{R}^{n-2} \times \Sigma$ , where  $\Sigma$  is diffeomorphic to the 2-sphere.*

The splitting in Theorem 1.2 ceases to be true if  $\text{scal}$  attains negative values, as the examples constructed by Sekigawa [36] show. Recently, a complete description of the metrics on complete simply-connected locally irreducible 3-manifolds with constant 0-nullity 1 and constant negative scalar curvature, as well as the topology of their quotients in case the fundamental group is finitely generated, has been obtained [8]. In view of Theorem 1.2 and [7, Thm. 1], the following question seems interesting:

**Question 1.1** *Is there a simply-connected complete irreducible Riemannian  $n$ -manifold with constant 0-conullity 2 and nonnegative sectional curvature?*

**Table 1** Left-invariant Riemannian metrics on 3-dimensional Lie groups with nontrivial (+1)-nullity

$\lambda_1$	$\lambda_2$	$\lambda_3$	$M$	$\text{scal}$	$\varphi$ -sect curv	Condition
$\theta + 1/\theta$	$\theta$	$1/\theta$	$SU(2)$	2	-1	$\theta > 0$
			$SU(2)$			$\theta > 0$
2	$\theta$	$\theta$	$\widetilde{SL}(2, \mathbb{R})$	$-2+4\theta$	$-3+2\theta$	$\theta < 0$
			$\text{Nil}^3$			$\theta=0$

**Table 2** Left-invariant Riemannian metrics on 3-dimensional Lie groups with nontrivial  $(-1)$ -nullity

$\lambda_1$	$\lambda_2$	$\lambda_3$	$M$	scal	$K_D$	Condition
$\theta - 1/\theta$	$-1/\theta$	$\theta$	$\widetilde{SL(2, \mathbb{R})}$ $E(1,1)$	-2	1	$0 < \theta < 1$ $\theta = 1$

In [7, Thm. 3] it was claimed that complete 4-manifolds with 0-nullity 1, non-zero splitting tensor, and finite volume do not exist. Whereas we believe this statement to be true, we found a fatal slip in the calculations in the proof. To compare, in dimension one higher we show:

**Theorem 1.3** *There exists a compact irreducible locally homogeneous Riemannian 5-manifold with 0-nullity 1.*

The manifold in Theorem 1.3 is in fact an almost Abelian Lie group. The idea of construction follows the example in [14, §9]. Other examples of compact locally homogeneous spaces with non-trivial nullity are given in [15].

### 1.3 Results with $\kappa = -1$

In the 3-dimensional case, the following result is closely related to [37, Thm. 1.1].

**Theorem 1.4** *Let  $M$  be a Riemannian  $n$ -manifold ( $n \geq 3$ ) with maximal  $(-1)$ -conullity 2.*

- (a) *If the scalar curvature is constant and  $\mathcal{D} = \mathcal{N}_{-1}^\perp$  is integrable on an open subset  $U$  of  $(-1)$ -conullity 2 then  $U$  is locally isometric to the group of rigid motions of the Minkowski plane,  $E(1, 1) = SO_0(1, 1) \ltimes \mathbb{R}^2$ , with a left-invariant metric.*
- (b) *Assume  $M$  is complete and has finite volume. Assume, in addition, that either  $n = 3$  or the scalar curvature is bounded away from  $-n(n - 1)$  (that is  $|\text{scal} + n(n - 1)|$  is bounded away from zero). Then the universal covering of  $M$  is homogeneous.*
- (c) *If  $M$  is homogeneous and simply-connected, then  $M$  is isometric to  $E(1, 1)$  or  $\widetilde{SL(2, \mathbb{R})}$  with a left-invariant metric.*

The left-invariant metrics in Theorem 1.4(c) can also be described following [26]. The groups listed are unimodular and we use the above notation to write (1.1). By switching  $e_2$  and  $e_3$ , and changing  $e_2$  to its opposite, if necessary, we may assume  $0 < \lambda_3 \leq 1$ , and then the possibilities are given in Table 2 ( $K_D$  denotes the sectional curvature of the 2-plane orthogonal to  $\mathcal{N}_{-1}$ ).

Note that  $SL(2, \mathbb{R})$  and  $E(1, 1)$  are both modelled on the same algebraic curvature tensor.

The following theorem deals with a situation of least non-trivial nullity. We may assume  $n \geq 4$  as the case  $n = 3$  is covered by Theorem 1.4.

**Theorem 1.5** *Let  $M$  be a complete Riemannian  $n$ -manifold ( $n \geq 4$ ) with constant  $(-1)$ -nullity 1 and finite volume. Then  $n$  is odd and the universal Riemannian covering of  $M$  is isometric to the solvable (almost-Abelian, unimodular) Lie group  $G = \mathbb{R} \ltimes \mathbb{R}^m$  ( $m = n - 1$ ), where  $\mathbb{R}^m$  is Abelian, and the adjoint action of a certain element of  $\mathbb{R}$  on  $\mathbb{R}^m$  is given in an orthonormal basis by the matrix  $\begin{pmatrix} I_{m/2} & 0 \\ 0 & -I_{m/2} \end{pmatrix}$*

According to formulae (4.17) below, if we replace the matrix in the statement of Theorem 1.5 by  $\begin{pmatrix} I_k & 0 \\ 0 & -I_{m-k} \end{pmatrix}$ , where  $k = 1, \dots, m - 1$ , we get a homogeneous (hence complete) Riemannian  $n$ -manifold  $\mathbb{R} \ltimes \mathbb{R}^m$  with constant  $(-1)$ -nullity 1, but it will not have quotients of finite volume, as it will not be unimodular, unless  $k = m/2$ .

## 1.4 Complete non-integrability of the conullity distribution

In the last part of this work, we give a simpler and unified proof of the following results due to Vittone [43] and Di Scala, Olmos and Vittone [14].

**Theorem 1.6** *Let  $M$  be either:*

- (a) *a connected complete Riemannian manifold of with nonzero constant index of  $\kappa$ -nullity, where  $\kappa > 0$ ; or*
- (b) *a connected simply-connected irreducible homogeneous Riemannian manifold with nonzero index of 0-nullity.*

*Then any two points of  $M$  can be joined by a piecewise smooth curve which is orthogonal to the distribution of  $\kappa$ -nullity at smooth points.*

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## 2 Preliminaries

The splitting tensor of the nullity distribution was introduced by Rosenthal in [32] under the name 'conullity operator'; it plays a key role in this work. Let  $M$  be a connected Riemannian manifold, and let  $\mathcal{D}$  be a smooth distribution on  $M$ . Consider the orthogonal splitting  $TM = \mathcal{D} \oplus \mathcal{D}^\perp$ . It will be convenient to call the  $\mathcal{D}$ -component (resp.  $\mathcal{D}^\perp$ -component) of tangent vectors the *horizontal* (resp. *vertical*) component, and, for a vector field  $X \in \Gamma(TM)$ , we shall write  $X = X^h + X^v$ . Now we can define the *splitting tensor* of  $\mathcal{D}^\perp$  as the map

$$C : \Gamma(\mathcal{D}^\perp) \times \Gamma(\mathcal{D}) \rightarrow \Gamma(\mathcal{D})$$

given by

$$C(T, X) = -(\nabla_X T)^h = C_T X$$

(see [16, p. 186]). It is clear that  $C$  is  $C^\infty(M)$ -linear in each variable. Note that in case  $\mathcal{D}$  is integrable,  $C$  is nothing but the shape operator of the leaves; further, this is the case if and only if  $C_{T_p} : \mathcal{D}_p \rightarrow \mathcal{D}_p$  is a symmetric endomorphism for all  $T \in \Gamma(\mathcal{D}^\perp)$  and all  $p \in M$ , since

$$\begin{aligned} \langle C_T X, Y \rangle - \langle X, C_T Y \rangle &= -\langle \nabla_X T, Y \rangle + \langle X, \nabla_Y T \rangle \\ &= \langle T, \nabla_X Y \rangle - \langle \nabla_Y X, T \rangle \\ &= \langle T, [X, Y] \rangle, \end{aligned} \tag{2.2}$$

for all  $X, Y \in \Gamma(\mathcal{D})$ . Of course,  $C$  vanishes identically if and only if  $\mathcal{D}$  is autoparallel.

In the remainder of this section, we assume that  $v_\kappa(p) > 0$  for some  $\kappa \in \mathbb{R}$  and for all  $p \in M$ , and we let  $\Delta \subset \mathcal{N}_\kappa$  be a nontrivial autoparallel distribution. In [33, Lem., p. 474] the following Riccati-type ODE for the splitting tensor is used (see also [17, Lem. 1]).

**Proposition 2.1** *The splitting tensor  $C$  of  $\Delta$  satisfies*

$$\nabla_T C_S = C_S C_T + C_{\nabla_T S} + \kappa \langle T, S \rangle I \tag{2.3}$$

for all  $S, T \in \Gamma(\Delta)$ . In particular, the operator  $C_{\gamma'}$ , along a unit speed geodesic  $\gamma$  in a leaf of  $\Delta$ , satisfies

$$(C_{\gamma'})' = C_{\gamma'}^2 + \kappa I, \quad (2.4)$$

where the prime denotes covariant differentiation along  $\gamma$ .

In general, we shall use the name  $\kappa$ -nullity geodesic to refer to a geodesic contained in a leaf of  $\kappa$ -nullity.

Following the ideas of [10], we can provide an explicit solution of equation (2.4).

**Proposition 2.2** *Let  $\gamma : [0, b) \rightarrow M$  be a nontrivial unit speed geodesic with  $p = \gamma(0)$  and  $\gamma'(0) \in \Delta_p$  so that  $\gamma$  is a geodesic of the leaf of  $\Delta$  through  $p$ . Assume that  $\gamma([0, b))$  is contained in an open subset of  $M$  where  $v_\kappa$  is constant. Then the splitting tensor  $C_{\gamma'(t)} = C(t)$  of  $\Delta$  at  $\gamma(t)$  is given, in a parallel frame along  $\gamma$ , by*

$$C(t) = -J_0'(t)J_0(t)^{-1}, \quad (2.5)$$

where

$$J_0(t) = \begin{cases} \cos(\sqrt{\kappa}t)I - \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}}C_0 & \text{if } \kappa > 0, \\ \cosh(\sqrt{-\kappa}t)I - \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}}C_0 & \text{if } \kappa < 0, \\ I - tC_0 & \text{if } \kappa = 0, \end{cases} \quad (2.6)$$

and  $C_0 = C(0)$ . In particular  $J_0(t)$  is invertible for  $t \in [0, b)$ .

**Proof** The formula (2.5) is obtained by integration of (2.4). To see that  $J_0(t)$  is invertible for  $t \in [0, b)$ , note that  $J_0$  is the solution of the Jacobi equation  $J'' + \kappa J = 0$ , and so is the vector field  $U$  along  $\gamma$  given by the solution of  $U' + C(t)U = 0$  with initial condition  $U(0) = I$ . Note that  $J_0$  and  $U$  have the same initial conditions at  $t = 0$ , so they coincide. On the other hand,  $U$  satisfies a first order differential equation from which one easily sees that  $\ker U$  is parallel along  $\gamma$ . It follows that  $J_0(t) = U(t)$  is invertible for  $t \in [0, b)$ .  $\square$

Recall that the maximum number of linearly independent smooth vector fields on  $S^{m-1}$  is given by  $\rho(m) - 1$ , where  $\rho(m)$  is the  $m$ th Radon-Hurwitz number, defined as  $2^c + 8d$ , where  $m = (\text{odd})2^{c+4d}$  for  $d \geq 0$  and  $0 \leq c \leq 3$ . The invertibility of  $J_0(t)$  in Proposition 2.2 implies:

**Corollary 2.1** *Let  $\gamma : [0, b) \rightarrow M$  be as in Proposition 2.2, where  $b = \infty$ .*

- (a) *If  $\kappa > 0$ , then the splitting tensor  $C_{\gamma'}$  has no real eigenvalues. It follows that  $\rho(n-d) \geq d+1$ , where  $n = \dim M$  and  $d = \dim \Delta$*
- (b) *If  $\kappa \leq 0$ , then any real eigenvalue  $\lambda$  of  $C_{\gamma'}$  satisfies  $|\lambda| \leq \sqrt{-\kappa}$ .*

**Proof** The assertion about the Radon-Hurwitz number in (a) goes as follows (cf. [17, Thm. 1]). Fix an orthonormal basis  $T_1, \dots, T_d$  of  $\Delta$ . For every unit  $X \in \Delta^\perp$ , the list  $X, C_{T_1}X, \dots, C_{T_d}X$  must be linearly independent, for otherwise  $C_T$  would have a real eigenvalue for some  $T \in \Delta$ . Now  $C_{T_1}X, \dots, C_{T_d}X$  projects to a global frame on the unit sphere of  $\Delta^\perp$ .  $\square$

Recall that a smooth distribution  $\mathcal{D}$  on  $M$  is called *bracket-generating* if the iterated Lie brackets of smooth sections of  $\mathcal{D}$  eventually span the whole  $TM$ . More precisely, we identify  $\mathcal{D}$  with its sheaf of smooth local sections, and define  $\mathcal{D}^1 = \mathcal{D}$  and

$$\mathcal{D}^{r+1} = \mathcal{D}^r + [\mathcal{D}, \mathcal{D}^r] = [\mathcal{D}, \mathcal{D}^r] \quad (2.7)$$

for  $r \geq 1$ , where

$$[\mathcal{D}, \mathcal{D}^r] = \{[X, Y] : X \in \mathcal{D}, Y \in \mathcal{D}^r\}.$$

Note that  $\mathcal{D}^r$  for  $r \geq 2$  in general has variable rank. We say that  $\mathcal{D}$  is *bracket-generating of step  $r$*  if, for some  $r \geq 2$ , we have  $\mathcal{D}^r = TM$  and  $r$  is the minimal integer satisfying this condition.

**Corollary 2.2** *If  $M$  is complete,  $\kappa > 0$  and  $v_\kappa$  is constant, then  $\mathcal{D} := \mathcal{N}_\kappa^\perp$  is bracket-generating of step 2.*

**Proof** The calculation (2.2) shows that  $C_{T_p}$  is symmetric for  $T_p \perp \mathcal{D}_p^2$  and  $p \in M$ , and thus has all eigenvalues real. Now Corollary 2.1 implies  $T_p = 0$ .  $\square$

**Remark 2.1** The above arguments also easily imply Theorem 4 in [34], which states that for a compact  $2n + 1$ -dimensional Sasakian manifold  $M$  with constant  $v_\kappa > 0$  for some  $\kappa > 0$  either  $v_\kappa \leq n$  or  $M$  has constant curvature  $\kappa$ . Indeed, assume  $M$  has not constant curvature and apply Corollary 2.1(a) to it to obtain

$$\rho(2n + 1 - v_\kappa) \geq v_\kappa + 1.$$

Now the desired result immediately follows from the trivial estimate  $m \geq \rho(m)$  for all  $m$ .

The case  $\kappa = 0$  of (2.9) below can be found in [6, ch. 4, §4.1].

**Lemma 2.1** *Let  $\gamma : [0, b) \rightarrow M$  with  $\gamma' = T \in \Delta$  be as in Proposition 2.2. Denote the scalar curvature of  $M$  by  $\text{scal}$ , and put  $n = \dim M$  and  $d = \dim \Delta$ . Then*

$$\frac{1}{2} \frac{d}{dt} \text{scal} = -\kappa(n - d - 1) \text{tr } C_T + \sum_{i \neq j} \langle R(C_T X_i, X_j) X_j, X_i \rangle, \quad (2.8)$$

where  $\{X_i\}_{i=1}^{n-d}$  is a parallel orthonormal frame of  $\Delta^\perp$  along  $\gamma$ .

In particular, in case  $\Delta = \mathcal{N}_\kappa$ , and  $v_\kappa = n - 2$  along  $\gamma$ , we have

$$\frac{1}{2} \frac{d}{dt} \text{scal} = \text{tr } C_T (K_{\mathcal{D}} - \kappa), \quad (2.9)$$

where  $K_{\mathcal{D}}$  denotes the sectional curvature of the 2-plane distribution  $\mathcal{D} = \mathcal{N}^\perp$ .

Further, if in addition  $\text{scal}$  is constant, then  $\text{tr } C_T = 0$  and  $\det C_T = \kappa$  along  $\gamma$ .

**Proof** Let  $\{T = T_1, T_2, \dots, T_d\}$  be an orthonormal frame of  $\Delta$  which is parallel along  $\gamma$ . We compute

$$\text{scal} = \sum_{i \neq j} \langle R(T_i, T_j) T_j, T_i \rangle + \sum_{i,j} \langle R(X_i, T_j) T_j, X_i \rangle + \sum_{i \neq j} \langle R(X_i, X_j) X_j, X_i \rangle.$$

Since the first two sums on the right-hand side are constant along  $\gamma$ , we get

$$\begin{aligned} \frac{d}{dt} \text{scal} &= \sum_{i \neq j} \langle \nabla_T R(X_i, X_j) X_j, X_i \rangle \\ &= - \sum_{i \neq j} \langle \nabla_{X_i} R(X_j, T) X_j, X_i \rangle + \langle \nabla_{X_j} R(T, X_i) X_j, X_i \rangle \\ &= -2 \sum_{i \neq j} \langle \nabla_{X_j} R(T, X_i) X_j, X_i \rangle, \end{aligned} \quad (2.10)$$

where we have used the Bianchi identity and other symmetries of  $R$ .

Next, since  $T \in \mathcal{N}_\kappa$ , we can write

$$\begin{aligned}\nabla_{X_j} R(T, X_i) X_j &= -\nabla_{X_j}(\kappa(T \wedge X_i) X_j) - R(\nabla_{X_j} T, X_i) X_j \\ &\quad + \kappa(T \wedge \nabla_{X_j} X_i) X_j + \kappa(T \wedge X_i) \nabla_{X_j} X_j \\ &= \kappa(C_T X_j, X_i) X_j + R(C_T X_j, X_i) X_j.\end{aligned}$$

Substituting into (2.10) yields (2.8).

In case  $d = v_\kappa = n - 2$  we have

$$\begin{aligned}\langle R(C_T X_1, X_2) X_2, X_1 \rangle + \langle R(C_T X_2, X_1) X_1, X_2 \rangle &= \langle R(X_1, X_2) X_2, C_T X_1 \rangle \\ &\quad + \langle R(X_2, X_1) X_1, C_T X_2 \rangle \\ &= (\text{tr } C_T) K_{\mathcal{D}},\end{aligned}$$

and (2.9) follows.

Since  $v_\kappa = n - 2$  along  $\gamma$ , we have  $K_{\mathcal{D}} \neq \kappa$  along  $\gamma$ . Therefore, in case  $\text{scal}$  is constant, equation (2.9) yields  $\text{tr } C_T = 0$  along  $\gamma$ . Finally, take the trace in (2.4) and use the characteristic polynomial  $C_T^2 + (\det C_T)I = 0$  to obtain  $\det C_T = \kappa$  along  $\gamma$ .  $\square$

**Lemma 2.2** *Assume  $\kappa \leq 0$ ,  $\gamma$  is a complete  $\kappa$ -nullity geodesic,  $v_\kappa = n - 2$  and  $K_{\mathcal{D}}$  is bounded away from  $\kappa$  along  $\gamma$ . Then  $\text{tr } C(t) = 0$  and  $\det C(t) = \kappa$  for all  $t \in \mathbb{R}$ , where  $C(t) = C_{\gamma'(t)}$ .*

**Proof** Note that  $\frac{1}{2}\text{scal} = K_{\mathcal{D}} + m\kappa$ , where  $m = \frac{n^2-n}{2} - 1$ . Using equations (2.5), (2.9) and Jacobi's formula, we obtain

$$\begin{aligned}\frac{d}{dt}(K_{\mathcal{D}} - \kappa) &= \text{tr}(-J'_0 J_0^{-1})(K_{\mathcal{D}} - \kappa) \\ &= -\frac{\frac{d}{dt} \det J_0}{\det J_0}(K_{\mathcal{D}} - \kappa).\end{aligned}$$

Integration of this equation yields

$$K_{\mathcal{D}}(t) - \kappa = (K_{\mathcal{D}}(0) - \kappa) |\det J_0(t)|^{-1}.$$

Now  $\det J_0(t)$  equals

$$1 - (\text{tr } C_0)t + (\det C_0)t^2$$

where  $C_0 = C(0)$ , if  $\kappa = 0$ , and

$$\frac{1}{2} \left(1 + \frac{\det C_0}{\kappa}\right) + \frac{1}{4} \left(1 - \frac{\det C_0}{\kappa} - \frac{\text{tr } C_0}{\sqrt{-\kappa}}\right) e^{2t} + \frac{1}{4} \left(1 - \frac{\det C_0}{\kappa} + \frac{\text{tr } C_0}{\sqrt{-\kappa}}\right) e^{-2t}$$

if  $\kappa < 0$ . Since  $K_{\mathcal{D}}$  is bounded away from  $\kappa$ , and the initial point is arbitrary along  $\gamma$ , the desired result follows.  $\square$

**Lemma 2.3** *Let  $M$  be a complete Riemannian  $n$ -manifold of finite volume and minimal  $(-1)$ -nullity 1. Then  $\text{div } T = 0$ , where  $T$  is a unit vector field, tangent to the nullity, defined in the open set of minimal nullity.*

**Proof** Let  $\gamma$  be an integral curve of  $T$ , a complete unit speed  $(-1)$ -nullity geodesic, and put  $C(t) := C_{\gamma'(t)}$ . According to (2.5),

$$C(t) = (-\sinh tI + \cosh tC_0)(\cosh tI - \sinh tC_0)^{-1}, \quad (2.11)$$

where  $C_0 = C(0)$ . Now Jacobi's formula yields

$$\begin{aligned} \operatorname{tr} C(t) &= -\frac{\frac{d}{dt} \det(\cosh tI - \sinh tC_0)}{\det(\cosh tI - \sinh tC_0)} \\ &= -\frac{P(\xi)}{Q(\xi)}, \end{aligned}$$

where  $\xi = \tanh t$  and

$$P(\xi) = \sum_{j=0}^m (-1)^j [(m-j)\xi^{j+1} + j\xi^{j-1}] \sigma_j,$$

and

$$Q(\xi) = \sum_{j=0}^m (-1)^j \xi^j \sigma_j;$$

here  $m = n - 1$  and  $\sigma_j = \sigma_j(C_0)$  denotes the  $j$ -symmetric function of the eigenvalues of  $C_0$ . Note that  $Q$  is nothing but the characteristic polynomial of  $C_0$ .

In order to compute the limits of  $\operatorname{tr} C(t)$  as  $t \mapsto \pm\infty$ , note that if  $Q(1) = Q'(1) = \dots = Q^{(k-1)}(1) = 0$  for some  $k = 0, \dots, m$ , then the alternate sums

$$\sum_{j=0}^m (-1)^j \sigma_j = \sum_{j=0}^m (-1)^j j \sigma_j = \dots = \sum_{j=0}^m (-1)^j j^{k-1} \sigma_j = 0.$$

Therefore

$$\begin{aligned} P^{(k)}(1) &= \sum_{j=0}^m (-1)^j j(j-1) \dots (j-k+2) [(m-2k)j + m + k^2 - k] \sigma_j \\ &= \sum_{j=0}^m (-1)^j j(j-1) \dots (j-k+2) (m-2k)j \sigma_j \\ &= (m-2k) \sum_{j=0}^m (-1)^j j(j-1) \dots (j-k+2) (j-k+1) \sigma_j \\ &= (m-2k) Q^{(k)}(1), \end{aligned}$$

and L'Hôpital rule yields

$$\lim_{t \rightarrow +\infty} \operatorname{tr} C(t) = \lim_{\xi \rightarrow 1} -\frac{P(\xi)}{Q(\xi)} = -(m-2k).$$

In a similar vein, if  $-1$  is a root of multiplicity  $k$  of the polynomial  $Q$ , we compute that  $\lim_{t \rightarrow -\infty} \operatorname{tr} C(t) = m - 2k$ .

Next, we use the above calculation to prove the following claim: the eigenvalues of  $C_0$  are  $-1$  and  $+1$ , each with multiplicity  $m/2$ . In particular,  $m$  is even and the divergence  $\operatorname{div} T = \operatorname{tr} \nabla T = -\operatorname{tr} C_T = 0$  everywhere.

Suppose the claim is not true at  $p \in M$ . Let  $\gamma : \mathbb{R} \rightarrow M$  be a nullity geodesic with  $\gamma(0) = p$ ,  $C(t) := C_{\gamma'(t)}$ . Now

$$\lim_{t \rightarrow +\infty} \operatorname{tr} C(t) = -(m - 2k_+), \quad \lim_{t \rightarrow -\infty} \operatorname{tr} C(t) = m - 2k_-,$$

where  $k_{\pm}$  is the multiplicity of  $\pm 1$  as an eigenvalue of  $C_0$ . If  $k_+ < m/2$  then  $\lim_{t \rightarrow +\infty} \operatorname{div} T = m - 2k_+ > 0$ . Since  $k_+$  is an upper semicontinuous function, we have  $m - 2k_+ > 0$  on a neighborhood of  $p$  in  $M$ . Now we can find a (compact)  $m$ -disk transversal to  $\mathcal{N}_{-1}$ , containing  $p$  in its interior, and  $t_0, L > 0$  such that  $\operatorname{div} T|_{\gamma_x(t)} > L$  for all  $t \geq t_0$  and  $x \in D$ ; here  $\gamma_x$  denotes the nullity geodesic with  $\gamma_x(0) = x$ ,  $\gamma'_x(0) = T_x$ . Put

$$U(t) := \{\gamma_x(s) \mid x \in D, s \geq t\}, \quad v_s := \operatorname{vol}(U(t_0 + s)).$$

Note that  $v_s > 0$  since  $U(t)$  has non-empty interior, and  $v_s < \infty$  by our assumption. For  $0 \leq s_1 < s_2$  we have  $U(t_0 + s_2) \subset U(t_0 + s_1)$  and thus  $v_{s_2} \leq v_{s_1}$ . On the other hand, the Divergence Theorem and the First Variation of Volume imply

$$\frac{d}{ds} v_s = \int_{U(t_0+s)} \operatorname{div} T > 0,$$

a contradiction. This proves that  $k_+ \geq m/2$ .

If  $k_- < m/2$ , we replace  $T$  by  $-T$ , so that  $C_0$  is replaced by  $-C_0$  and  $k_+$  and  $k_-$  are interchanged. Now  $k_+ < m/2$  and the argument above leads to a contradiction. Hence  $k_- \geq m/2$ . Since  $k_+ + k_- \leq m$ , we finally deduce that  $k_+ = k_- = m/2$ .

□

### 3 Manifolds with $\kappa$ -conullity 2

In this section, we obtain results in case  $M$  has maximal  $\kappa$ -conullity 2.

#### 3.1 The case $\kappa = 1$

We now prove Theorem 1.1. Note that, owing to Corollary 2.1(a),  $2 \geq \rho(2) \geq (n - 2) + 1$ , so  $n = 3$  and  $v_1 = 1$ .

For any  $T \in \mathcal{N}_1$ ,  $C_T$  is a  $2 \times 2$  real matrix without real eigenvalues, again by Corollary 2.1(a), thus with a pair of complex conjugate eigenvalues. Moreover Lemma 2.1 says  $\operatorname{tr} C_T = 0$  and  $\det C_T = 1$  if  $\|T\| = 1$ , so that the eigenvalues of  $C_T$  must be  $\pm i$ . Since  $C_T^2 = -I$ , equation (2.4) implies that  $C_{\gamma'}$  is constant along a unit speed nullity geodesic  $\gamma$  with respect to any parallel orthonormal frame of  $\mathcal{D} := \mathcal{N}_1^{\perp}$ , therefore we can write  $C_T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , for  $T = \gamma'$ , with respect to a parallel orthonormal frame of  $\mathcal{D}$  along  $\gamma$ .

Since  $C_T$  is skew-symmetric, the distribution  $\mathcal{D}$  is non-integrable. A nowhere integrable rank 2 distribution in a 3-manifold must be a contact distribution. For  $X \in \mathcal{D}$ , we have

$$\langle L_T X, T \rangle = \langle -\nabla_X T, T \rangle = -\frac{1}{2} X \cdot \|T\|^2 = 0.$$

Now the flow of  $T$  preserves  $\mathcal{D}$ , so  $T$  is the Reeb (or characteristic) vector field of  $\mathcal{D}$  [4, § 3.1]. Further,  $\nabla T = -C_T$  is skew-symmetric, so  $T$  is a Killing field. This says  $M$  is a  $K$ -contact distribution [4, § 6.2], which in dimension 3 is equivalent to Sasakian [4, Cor. 6.5].

A 3-dimensional Sasakian manifold with constant scalar curvature is locally  $\varphi$ -symmetric [44, Thm. 4.1]. Since  $M$  is assumed complete and simply-connected, it is a globally  $\varphi$ -

symmetric space [42, Thm. 6.2]. By the classification of Sasakian globally  $\varphi$ -symmetric spaces in dimension 3 [9, Thm. 11], we finally deduce that  $M$  is a Sasakian space form, that is, those listed in the statement of Theorem 1.1.

### 3.2 The case $\kappa = 0$

Now we deal with Theorem 1.2.

Since  $\text{scal} \neq 0$  everywhere, the conullity equals 2 everywhere. For each  $p \in M$ , consider the linear map  $C_p : \mathcal{N}_0|_p \rightarrow M(2, \mathbb{R})$ . Lemma 2.2 says that  $\text{tr } C_S = 0$  and  $\det C_S = 0$  for  $S \in \mathcal{N}_0$ , so the image of  $C_p$  is at most one-dimensional. Let  $U$  be the set of points  $p \in M$  such that  $C_p \neq 0$ . On  $U$  we choose a unit vector field  $T \in \mathcal{N}_0$  spanning the orthogonal complement to  $\ker C$  in  $\mathcal{N}_0$ . It follows from equation (2.3) that  $\nabla_T S \in \ker C$  for all  $S \in \ker C$ . Therefore  $\nabla_T T = 0$ .

Since  $\det C_T = 0$  and the real eigenvalues of  $C_T$  can only be zero, due to Corollary 2.1(b), the endomorphism  $C_T$  is nilpotent. Now for each  $p \in U$  we can find an orthonormal basis  $X_p, Y_p$  of  $\mathcal{D}_p = \mathcal{N}_0^\perp|_p$  such that  $C_T X_p = 0$  and  $C_T Y_p = a(p)X_p$  for some  $a(p) \neq 0$ ; on a connected component of  $U$ , we may assume  $a(p) > 0$  for all  $p$ . Also, it follows from equation (2.4) that  $X$  and  $Y$  can be taken parallel along a nullity geodesic, and then also the function  $a$  is constant along  $\gamma$ . By passing to the Riemannian universal covering of  $U$ , if necessary, we may define the orthonormal frame  $X, Y$  of  $\mathcal{D}$  globally. Now the Levi-Civita connection satisfies:

$$\begin{aligned} \nabla_T T &= \nabla_T X = \nabla_T Y = 0, \\ \nabla_X T &= (\nabla_X T)^v \perp T, \quad \nabla_Y T = -aX + (\nabla_Y T)^v, \\ \nabla_X X &= \alpha Y, \quad \nabla_Y Y = \beta X, \quad \nabla_X Y = -\alpha X, \quad \nabla_Y X = -\beta Y + aT, \end{aligned}$$

for some smooth functions  $\alpha, \beta$  on  $U$ . We compute that

$$R(X, Y)X = (X(\alpha) - a\beta)T + (\alpha^2 + \beta^2 - X(\beta) - Y(\alpha))Y + a(\nabla_X T)^v,$$

and

$$R(Y, X)Y = -a\alpha T + (\alpha^2 + \beta^2 - X(\beta) - Y(\alpha))X.$$

From  $T \in \mathcal{N}_0$  we deduce that

$$\alpha = 0, \quad X(\alpha) = a\beta, \quad \text{scal} = 2(X(\beta) - \beta^2). \quad (3.12)$$

In particular any integral curve  $\eta$  of  $X$  in  $U$  is a geodesic. By completeness of  $M$ , the curve  $\eta$  can be extended to a complete geodesic. We claim that  $\eta$  is entirely contained in  $U$ . Indeed the second equation (3.12) yields that

$$\frac{d}{dt} \log a(\eta(t)) = \beta(\eta(t)),$$

and hence

$$a(\eta(t)) = a(\eta(0))e^{\int_0^t \beta(\eta(\xi)) d\xi}.$$

The third equation in (3.12) says that  $X(\beta) = \frac{1}{2}\text{scal} + \beta^2 > 0$ , so

$$a(\eta(t)) \geq a(\eta(0))e^{t\beta(\eta(0))} > 0$$

for  $t > 0$ . In particular  $a$  is bounded away from zero along  $\eta$  for positive time. Repeating the argument for negative time yields that  $\eta$  is contained in  $U$ . By assumption  $\text{scal} \geq 2\delta^2$  for some  $\delta > 0$ , so  $X(\beta) \geq \delta^2 + \beta^2$ . After integration, we can write

$$\arctan(\delta^{-1}\beta(\eta(t))) \geq \delta t + \arctan(\delta^{-1}\beta(\eta(0)))$$

for all  $t \in \mathbb{R}$ . This is a contradiction, since the right-hand side is unbounded. Hence  $U = \emptyset$ , which is to say  $C \equiv 0$ , and this implies that  $M$  splits.

### 3.3 The case $\kappa = -1$

In this subsection, we prove Theorem 1.4.

We will first consider parts (a) and (c) of the statement. Note that the scalar curvature is constant under the assumptions there. For each  $p \in M$ , consider the linear map  $C_p : \mathcal{N}_{-1}|_p \rightarrow M(2, \mathbb{R})$ . Since the scalar curvature is constant, Lemma 2.1 says that  $\text{tr } C_T = 0$  and  $\det C_T = -1$  for unit  $T \in \mathcal{N}_{-1}$ , so  $C_p$  is injective and its image lies in the 3-dimensional subspace of traceless matrices. Moreover,  $C_p$  cannot be onto the subspace of traceless matrices, as this subspace contains singular matrices. Therefore  $\dim \mathcal{N}_{-1}|_p < 3$ , and hence  $n < 5$ .

We next rule out the case  $n = 4$ . By dimensional arguments the image of  $C_p$  meets the subspace of symmetric endomorphisms of  $\mathcal{D}_p$ , for each  $p \in M$ . Take the trace of equation (2.3) throughout, and use that  $\text{tr } C_T = 0$  for all  $T \in \mathcal{N}_{-1}$  and that the trace commutes with the covariant derivative, to obtain

$$\text{tr}(C_S C_T) = 2 \langle T, S \rangle$$

for all  $S, T \in \mathcal{N}_{-1}$ . Now we can find local orthonormal frames  $T_1, T_2$  of  $\mathcal{N}_{-1}$  and  $X, Y$  of  $\mathcal{D}$  such that  $C_{T_1}$  and  $C_{T_2}$  are respectively represented by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix},$$

where  $b$  is a nowhere zero locally defined smooth function on  $M$ . We refer again to equation (2.3) to write

$$\nabla_{T_2} C_{T_1} = C_{T_1} C_{T_2} + C_{\nabla_{T_2} T_1},$$

and identify the endomorphisms with their matrices to obtain

$$0 = \begin{pmatrix} 0 & b \\ -1/b & 0 \end{pmatrix} + \langle \nabla_{T_2} T_1, T_2 \rangle \begin{pmatrix} 0 & b \\ 1/b & 0 \end{pmatrix},$$

which clearly is impossible.

Now  $n = 3$ . Let  $T \in \mathcal{N}_{-1}$ ,  $\|T\| = 1$ . We already know endomorphisms with their matrices to obtain endomorphisms with their matrices to obtain endomorphisms with their matrices to obtain that  $\text{tr } C_T = 0$  and  $\det C_T = -1$ , so the eigenvalues of  $C_T$  are  $\pm 1$ . Since  $C_T^2 = I$ , equation (2.4) implies that  $C_{\gamma'}$  is constant along a nullity geodesic  $\gamma$  with respect to any parallel orthonormal frame of  $\mathcal{D} := \mathcal{N}_{-1}^\perp$ . Then we can write  $C_T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  along  $T = \gamma'$  with respect to a parallel frame of unit vector fields  $\tilde{X}, \tilde{Y}$  of  $\mathcal{D}$  along  $\gamma$ . Note that this frame is orthogonal at  $p$  if and only if  $C_{T_p}$  is a symmetric endomorphism if and only if  $\mathcal{D}$  is integrable at  $p$ .

In any case we have a locally defined frame  $T, \tilde{X}, \tilde{Y}$ , where we put  $f := -\langle \tilde{X}, \tilde{Y} \rangle$ , and we orthonormalize it to get

$$X = \frac{1}{\sqrt{1-f^2}}(\tilde{X} + f\tilde{Y}), \\ Y = \tilde{Y}.$$

With respect to  $X, Y$  we have

$$C_T = \begin{pmatrix} -1 & 0 \\ 2F & 1 \end{pmatrix},$$

where we have set  $F := f/\sqrt{1-f^2}$ . Note that  $f$  is constant along  $\gamma$ , so  $X, Y$  are parallel along  $\gamma$  and  $T(F) = 0$ . Hence we can write the Levi-Civita connection as follows:

$$\begin{aligned} \nabla_T T &= \nabla_T X = \nabla_T Y = 0, \quad \nabla_X T = X - 2FY, \quad \nabla_Y T = -Y, \\ \nabla_X X &= -T + \alpha Y, \quad \nabla_Y Y = T + \beta X, \quad \nabla_X Y = 2FT - \alpha X, \quad \nabla_Y X = -\beta Y, \end{aligned} \quad (3.13)$$

for some locally defined smooth functions  $\alpha, \beta$ . The bracket relations follow:

$$[X, Y] = 2FT - \alpha X + \beta Y, \quad [T, X] = -X + 2FY, \quad [T, Y] = Y. \quad (3.14)$$

Next, the curvature relations

$$\langle R(X, Y)X, Y \rangle = -K_{\mathcal{D}},$$

where  $K_{\mathcal{D}}$  is the sectional curvature of the plane spanned by  $X, Y$ , and

$$\langle R(X, Y)X, T \rangle = \langle R(T, Y)X, Y \rangle = \langle R(X, Y)Y, T \rangle = 0,$$

yield the equations

$$\begin{aligned} \alpha &= -\beta F, \\ T(\beta) &= \beta, \\ Y(F) &= -\beta(1+F^2), \\ X(\beta) - FY(\beta) &= K_{\mathcal{D}} - 1. \end{aligned} \quad (3.15)$$

With these equations at hand, we can finish the proofs of parts (a) and (c). In view of (3.14),  $\mathcal{D}$  is integrable on an open set  $U$  if and only if  $F$  vanishes identically on  $U$ . Assume this is the case. Equations (3.15) then imply  $\alpha = \beta = 0$  and  $K_{\mathcal{D}} = 1$ . Now (3.14) reduces to

$$[X, Y] = 0, \quad [T, X] = -X, \quad [T, Y] = Y.$$

In other words, we have a local orthonormal frame of vector fields whose Lie brackets have constant coefficients in this frame. Owing to Lie's third fundamental theorem, (see also [21, (1.4)] or [45, Lem. 2.5]),  $U$  is locally isometric to a Lie group with left-invariant metric; in this case,  $E(1, 1) = SO_0(1, 1) \ltimes \mathbb{R}^2$ . This proves part (a).

Assume now  $M$  is simply-connected and homogeneous as in (c). Then  $F$  is constant, and the equations (3.15) say that, again,  $\alpha = \beta = 0$  and  $K_{\mathcal{D}} = 1$ . Lie's third fundamental theorem yields that  $M$  is isometric to  $E(1, 1)$  in case  $F = 0$ , and to  $\widetilde{SL(2, \mathbb{R})}$  in case  $F \neq 0$ . This proves (c).

Finally, we deal with (b). We assume  $M$  is complete and has finite volume. Fix a complete unit speed nullity geodesic  $\gamma$  in the open set of minimal  $(-1)$ -nullity  $\Omega$ . If the scalar curvature of  $M$  is bounded away from  $-n(n-1)$ , then we apply Lemma 2.2 to deduce that the splitting

tensor  $C_{\gamma'(t)} = C(t)$  satisfies  $\text{tr } C(t) = 0$  and  $\det C(t) = -1$  for all  $t \in \mathbb{R}$ , and the argument follows as in the beginning of the proof to show that we must have  $n = 3$ . Then we can construct the locally defined orthonormal frame  $T, X, Y$  as above, and we have equations (3.15). Note that  $M$  has  $(-1)$ -nullity 1, so Lemma 2.3 says that  $\text{div } T = 0$ .

We follow an argument in [37]. Let  $\gamma$  be again a complete  $(-1)$ -nullity geodesic in  $\Omega$ , parameterized so that  $T = \gamma'$  along  $\gamma$ . The second equation in (3.15) implies that

$$\beta(\gamma(t)) = \beta(\gamma(0))e^t. \quad (3.16)$$

Let  $D$  be a compact 2-disk transversal to  $\gamma$  at  $\gamma(0)$ . By our assumption that the volume is finite and the Poincaré recurrence theorem,  $\gamma$  meets  $D$  infinitely many times as  $t \mapsto \pm\infty$ . Since  $\beta$  is bounded on  $D$ , equation (3.16) says that  $\beta$  vanishes identically along  $\gamma$ . Since  $\gamma$  is any nullity geodesic in  $\Omega$ , now  $\beta = 0$  on  $\Omega$ . The third equation in (3.15) gives  $Y(F) = 0$  and we also have  $T(F) = 0$ . Finally, apply the second bracket relation in (3.14) to  $F$  to get  $TX(F) = -X(F)$ . The same argument using the Poincaré recurrence theorem implies that  $X(F) = 0$ . Now  $\beta$  and  $F$  are constant on  $\Omega$ . In particular the scalar curvature is constant on the closure  $\bar{\Omega}$ , namely, equal to  $2(2\kappa + K_D) = -2$ . The complement  $M \setminus \Omega$  is a closed set consisting of isotropic points of  $M$ , namely, where all sectional curvatures are  $-1$  and hence the scalar curvature is  $-6$ . By connectedness of  $M$ ,  $\Omega = M$ . Now  $\beta$  and  $F$  are constant on  $M$  and this implies that the universal covering of  $M$  is homogeneous, via Lie's third fundamental theorem. This completes the proof of Theorem 1.4.

## 4 Almost Abelian Lie groups

An *almost Abelian* Lie group  $G$  is a non-Abelian (real connected) Lie group whose Lie algebra  $\mathfrak{g}$  has a codimension one Abelian ideal  $V$  (it is equivalent to require the existence of a codimension one subalgebra [2]). Hence we can write its Lie algebra as a semidirect product  $\mathfrak{g} = \mathbb{R} \ltimes_A V$ , where  $V$  is an Abelian ideal and the action of  $\mathbb{R}$  on  $V$  is determined by the adjoint action of a fixed generator  $\xi \in \mathbb{R}$ , which we represent by an operator  $A \in \mathfrak{gl}(V)$ , so that  $[\xi, X] = AX$  for all  $X \in V$ . Note that  $G = \mathbb{R} \ltimes_{e^A} V$ ,  $G$  is unimodular if and only if  $\text{tr } A = 0$ , and  $A \neq 0$  as  $G$  is non-Abelian.

First we compute the curvature of an almost Abelian Lie group  $G$ , equipped with a left-invariant Riemannian metric that makes  $\xi$  unit, and  $\xi$  and  $V$  orthogonal. Koszul's formula for the Levi-Civita connection immediately yields:

$$\nabla_\xi \xi = 0, \quad \nabla_\xi X = A^{sk} X, \quad \nabla_X \xi = -A^{sy} X, \quad \nabla_X Y = \langle A^{sy} X, Y \rangle \xi,$$

for all  $X, Y \in V$ , where  $A = A^{sy} + A^{sk}$  is the decomposition of  $A$  into its symmetric and skew-symmetric components. This easily gives expressions for the curvature tensor and the sectional curvatures as follows:

$$\begin{aligned} R(X, Y)Z &= -\langle A^{sy} Y, Z \rangle A^{sy} X + \langle A^{sy} X, Z \rangle A^{sy} Y, \\ R(X, Y)\xi &= 0, \\ R(\xi, X)Y &= \langle ([A^{sk}, A^{sy}] - (A^{sy})^2)X, Y \rangle \xi, \\ R(\xi, X)\xi &= ([A^{sy}, A^{sk}] + (A^{sy})^2)X, \end{aligned} \quad (4.17)$$

for  $X, Y, Z \in V$ . In particular, the sectional curvatures are given by:

$$K(\xi, X) = -\|A^{sy} X\|^2 - \langle [A^{sy}, A^{sk}]X, X \rangle,$$

and

$$K(X, Y) = -\det \begin{pmatrix} \langle A^{sy}X, X \rangle & \langle A^{sy}X, Y \rangle \\ \langle A^{sy}Y, X \rangle & \langle A^{sy}Y, Y \rangle \end{pmatrix},$$

for all  $X, Y \in V$ .

Let  $X_1, \dots, X_m$  be an orthonormal basis of  $V$  consisting of eigenvectors of  $A^{sy}$ , with corresponding eigenvalues  $\lambda_1, \dots, \lambda_m$ . Now we can express the Ricci curvature as:

$$Ric(\xi, \xi) = -\sum_{j=1}^m \lambda_j^2, \quad Ric(\xi, X_i) = 0,$$

$$Ric(X_i, X_i) = -\lambda_i \sum_{j=1}^m \lambda_j, \quad Ric(X_i, X_j) = (\lambda_i - \lambda_j) \langle A^{sk}X_i, X_j \rangle \quad (i \neq j).$$

Finally, the scalar curvature is

$$\text{scal} = -\sum_{i=1}^m \lambda_i^2 - \left( \sum_{i=1}^m \lambda_i \right)^2.$$

**Lemma 4.1** *If  $G = \mathbb{R} \ltimes_{e^A} V$  is not flat, then its 0-nullity distribution is the left-invariant distribution defined by the subspace*

$$\ker A^{sy} \cap (A^{sk})^{-1}(\ker A^{sy}). \quad (4.18)$$

of  $V$ .

**Proof** We use the notation above and formulae (4.17). Suppose  $a_0\xi + \sum_{i=1}^m a_i X_i \in \mathcal{N}_0$  for some  $a_0, a_1, \dots, a_m \in \mathbb{R}$ . The nonflatness assumption implies that  $A^{sy}$  is nonzero, so there is an index  $j$  such that  $\lambda_j \neq 0$ . Now

$$\begin{aligned} 0 &= R(a_0\xi + \sum_{i=1}^m a_i X_i, X_j)X_j \\ &= a_0 R(\xi, X_j)X_j + \sum_{i \neq j} a_i R(X_i, X_j)X_j \\ &= -a_0 \lambda_j^2 \xi - \lambda_j \sum_{i \neq j} a_i \lambda_i X_i. \end{aligned}$$

We deduce that  $a_0 = 0$  (so that  $\mathcal{N}_0 \subset V$ ), and  $a_i = 0$  for all  $i \neq j$  with  $\lambda_i \neq 0$ . If there is  $k \neq j$  with  $\lambda_k \neq 0$ , then we repeat the above argument with  $k$  in place of  $j$  to arrive at  $a_j = 0$ . If, on the contrary, all  $i \neq j$  have  $\lambda_i = 0$ , we use the following argument for  $X = \sum_i a_i X_i \in \mathcal{N}_0$ :

$$\begin{aligned} 0 &= R(\xi, X)\xi \\ &= A^{sy}A^{sk}X - A^{sk}A^{sy}X + (A^{sy})^2X \\ &= \langle A^{sk}X, X_j \rangle \lambda_j X_j - a_j \lambda_j A^{sk}X_j + a_j \lambda_j^2 X_j. \end{aligned}$$

Since  $A^{sk}X_j \perp X_j$ , the second term gives that  $a_j = 0$  or  $A^{sk}X_j = 0$ . In the latter case, the first term vanishes, and hence the last term gives again  $a_j = 0$ . In any case  $a_j = 0$ , so  $\mathcal{N}_0 \subset \ker A^{sy}$ . Finally, if  $X \in \mathcal{N}_0$  and  $Y \in V$  then

$$0 = \langle R(\xi, X)\xi, Y \rangle = \langle A^{sk}X, A^{sy}Y \rangle.$$

Therefore  $A^{sk} X \in (\text{im } A^{sy})^\perp = \ker A^{sy}$ , proving that  $\mathcal{N}_0$  is contained in (4.18). The converse inclusion is clear from (4.17), and this finishes the proof.  $\square$

We are now prepared to prove Theorem 1.3. Consider an almost Abelian group  $G = \mathbb{R} \ltimes_{e^A} V$ , where

$$A = \begin{pmatrix} 0 & -b & 0 & -c \\ b & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ c & 0 & 0 & a \end{pmatrix}$$

with respect to a basis  $X_1, \dots, X_4$  of  $V$ . Note that  $G$  is unimodular.

Take the left-invariant metric on  $G$  obtained by declaring the basis  $\xi, X_1, \dots, X_4$  orthonormal. Choose the coefficients  $a, b, c$  of the matrix  $A$  to be all nonzero. Thanks to Lemma 4.1, we immediately see that the nullity distribution is spanned by  $X_2$ . Now  $G$  has 0-nullity 1.

Suppose  $G$  splits as a Riemannian product. Then the conullity splits accordingly. It follows that either one of the factors is flat, or both have conullity 2. In the former case, one of the factors coincides with the 0-nullity, but this contradicts the fact that the splitting tensor  $C_{X_2}\xi = -(\nabla_\xi X_2)^h = -bX_1$  is nonzero. Therefore we must be in the latter case. The factors are of 0-conullity 2, therefore they are semi-symmetric. We note that the left-translations of  $G$  are isometries, and thus must preserve the factors. Now the factors are homogeneous; by [39, Prop. 5.1], they must be symmetric, and hence  $G$  is symmetric, a contradiction. Hence  $G$  is irreducible.

Finally,  $G$  has quotients of finite volume if, in addition,  $e^A$  can be represented by a matrix with integral coefficients in some basis of  $V$ , in view of the following result (see [18, Cor. 6.4.3]; here it is worth mentioning that every finite volume quotient of a solvable Lie group by a discrete subgroup is automatically compact, a result due to Mostow [28]).

**Lemma 4.2** (Filipkiewicz's criterion) *Suppose  $G = \mathbb{R} \ltimes_{e^A} V$  is unimodular and non-nilpotent. Then there is a discrete subgroup  $\Gamma$  of  $G$  with  $\Gamma \backslash G$  compact if and only if there exists  $\lambda \in \mathbb{R}$ ,  $\lambda \neq 0$ , such that  $e^{\lambda A}$  has a characteristic polynomial with integral coefficients.*

In order to find  $A$  satisfying the condition of Lemma 4.2, consider the standard  $4 \times 4$  matrix with two real eigenvalues and two complex conjugate ones:

$$B = \begin{pmatrix} -\gamma & 0 & 0 & 0 \\ 0 & \gamma - 2\alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}.$$

As claimed in [22] (see also [5, Prop. 3.7.2.5]), there exists  $\alpha, \beta, \gamma \in \mathbb{R}$  such that

$$e^B = \begin{pmatrix} e^{-\gamma} & 0 & 0 & 0 \\ 0 & e^{\gamma-2\alpha} & 0 & 0 \\ 0 & 0 & e^\alpha \cos \beta & -e^\alpha \sin \beta \\ 0 & 0 & e^\alpha \sin \beta & e^\alpha \cos \beta \end{pmatrix}$$

is conjugate to

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The approximate values are

$$\alpha \approx 0.308333405, \beta \approx 0.511773474, \gamma \approx 1.861109547.$$

Now it suffices to find  $a, b, c \in \mathbb{R} \setminus \{0\}$  such that  $A$  and  $B$  are conjugate. The eigenvalues of  $B$  are pairwise distinct, so it is enough to know that the characteristic polynomials  $p_A$  and  $p_B$  of  $A$  and  $B$ , resp., are equal.

We have

$$p_A(x) = x^4 + (-a^2 + b^2 + c^2)x^2 + ac^2x - a^2b^2,$$

and

$$p_B(x) = x^4 + \sigma x^2 + \mu x + \nu,$$

where

$$\begin{aligned} \sigma &= -2\alpha^2 + \beta^2 - (\alpha - \gamma)^2, & \mu &= 2\alpha((\alpha - \gamma)^2 + \beta^2), \\ \nu &= (\alpha^2 + \beta^2)\gamma(2\alpha - \gamma). \end{aligned}$$

Clearly  $\mu > 0$  and  $\nu < 0$ . It follows that we can solve  $a^4 + \sigma a^2 - \mu a + \nu = 0$  for real  $a > 0$ . Now  $a, b = \sqrt{-\nu}/a, c = \sqrt{\mu/a}$  yields a matrix  $A$  such that  $p_A = p_B$ . This completes the proof of Theorem 1.3.

## 5 Manifolds of $(-1)$ -nullity 1

In this section we prove Theorem 1.5.

By Lemma 2.3,  $\operatorname{div} T = 0$  for any unit vector field  $T$  in the nullity  $\mathcal{N}_{-1}$ . It also follows from the proof of that lemma that  $m = n - 1$  is even and the eigenvalues of  $C_T$  are  $-1$  and  $+1$ , each with multiplicity  $m/2$ . Fix  $p \in M$ . Since  $C_{T_p}$  has real eigenvalues, it is triangularizable over  $\mathbb{R}$ , hence it is triangularizable in an orthonormal basis. Choose an orthonormal basis of  $\mathcal{N}_{-1}^\perp|_p$  with respect to which the matrix of  $C_{T_p}$  is lower triangular. Let  $\gamma$  be a nullity geodesic with  $\gamma'(0) = T_p$ . Parallel translate that basis to an orthonormal frame along  $\gamma$ . Equation (2.11) shows that  $C(t) = C_{\gamma'(t)}$  is lower triangular in that frame, with eigenvalues  $\pm 1$ , each with multiplicity  $m/2$ , and off diagonal entries given by polynomials in  $e^t, e^{-t}$ . The assumption of finite volume together with the fact that  $T$  is divergence-free implies, via the Poincaré Recurrence Theorem, that  $\gamma$  must come back arbitrarily close to  $p$ , and infinitely often. We deduce that such off diagonal polynomials entries must be constant. Now  $0 = C'(t) = C(t)^2 - I$ , thanks to (2.4). In particular  $C := C(t) = C(0)$  is conjugate to a diagonal matrix with entries  $\pm 1$ , each repeated  $m/2$  times. Since  $C$  is diagonalizable,  $\dim \ker(C \pm I) = m/2$ , and so  $C$  has the block form

$$C = \begin{pmatrix} I_{m/2} & 0 \\ D & -I_{m/2} \end{pmatrix}. \quad (5.19)$$

Let  $T, X_1, \dots, X_m$  be a locally defined orthonormal frame of  $M$ , with respect to which  $C_T$  has the form (5.19), which is parallel along nullity geodesics. Then  $\nabla_T T = \nabla_T X_i = 0$ , and  $\nabla_{X_i} X_j = \sum_{k=1}^m \Gamma_{ij}^k X_k + \langle C_T X_i, X_j \rangle T$  for all  $i, j = 1, \dots, m$ . We claim that all “Christoffel symbols”  $\Gamma_{ij}^k$  ( $i, j, k = 1, \dots, m$ ) vanish.

In order to check the claim, we compute

$$R(T, X_m)X_i = \sum_k (T(\Gamma_{mi}^k) + \Gamma_{mi}^k)X_k$$

for  $i < m$  and

$$R(T, X_m)X_m = \sum_k (T(\Gamma_{mm}^k) + \Gamma_{mm}^k)X_k - T.$$

Using that  $T \in \mathcal{N}_{-1}$ , we get  $T(\Gamma_{mi}^k) = -\Gamma_{mi}^k$ . Poincaré recurrence now gives that  $\Gamma_{mi}^k = 0$  for  $i, k = 1, \dots, m$ . Next, assume by induction that  $\Gamma_{ij}^k = 0$  for  $i = i_0, \dots, m$ , for some  $i_0$ , and for  $j, k = 1, \dots, m$ . Since

$$[T, X_{i_0-1}] = C_T X_{i_0-1} = \pm X_{i_0-1} + \text{lin. comb. of } X_{i_0}, \dots, X_m,$$

we get

$$R(T, X_{i_0-1})X_j = \nabla_T \nabla_{X_{i_0-1}} X_j \pm \nabla_{X_{i_0-1}} X_j + \underbrace{\text{lin. comb. of } \nabla_{X_{i_0}} X_j, \dots, \nabla_{X_m} X_j}_{\text{multiple of } T},$$

so

$$0 = \langle R(T, X_{i_0-1})X_j, X_k \rangle = T(\Gamma_{i_0-1,j}^k) \pm \Gamma_{i_0-1,j}^k,$$

and we use Poincaré recurrence again to get  $\Gamma_{i_0-1,j}^k = 0$ . This proves that  $\Gamma_{ij}^k = 0$  for  $i, j, k = 1, \dots, m$ ; in particular,  $[X_i, X_j]$  can only have component in  $T$ , if any.

Let  $f$  be a locally defined smooth function on  $M$  representing an off diagonal entry of  $C_T$ . We already know that  $T(f) = 0$ . Now

$$TX_m(f) = [T, X_m](f) = C_T X_m(f) = -X_m(f),$$

so  $X_m(f) = 0$  by Poincaré recurrence. Suppose now, by induction, that  $X_m(f) = \dots = X_{i_0+1}(f) = 0$  for some  $i_0$ . Then

$$\begin{aligned} TX_{i_0}(f) &= [T, X_{i_0}](f) = C_T X_{i_0}(f) \\ &= \pm X_{i_0}(f) + \underbrace{(\text{lin. comb. of } X_{i_0+1}, \dots, X_m)(f)}_{=0}. \end{aligned} \tag{5.20}$$

Therefore  $X_{i_0}(f) = 0$ , by Poincaré recurrence. It follows that  $X_i(f) = 0$  for all  $i$  and hence  $f$  is locally constant. This already implies that the real vector space spanned by the locally defined frame  $T, X_1, \dots, X_m$  is closed under commutators, and hence  $M$  is locally isometric to a Lie group with a left-invariant metric, due to Lie's Third Theorem.

We identify the Lie group. Note that, going along directions other than  $\mathcal{N}_{-1}$ , the frame  $X_1, \dots, X_m$  is defined up to a transformation  $\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$ , where  $P \in O(m/2)$  (resp.  $Q \in O(m/2)$ ) acts on the  $+1$ -eigenspace of  $C_T$  (resp.  $-1$ -eigenspace of  $C_T$ ). On one hand, since we have shown the entries of  $C_T$  to be locally constant, the matrix (5.19) has  $D$  constant. On the other hand, (5.19) satisfies

$$\begin{pmatrix} P(t) & 0 \\ 0 & Q(t) \end{pmatrix} \begin{pmatrix} I & 0 \\ D & -I \end{pmatrix} \begin{pmatrix} P(t) & 0 \\ 0 & Q(t) \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ D & -I \end{pmatrix}$$

for all smooth curves  $P(t), Q(t) \in SO(m/2)$ , with  $P(0) = Q(0) = I_{m/2}$ , along a smooth curve  $c : (-\epsilon, \epsilon) \rightarrow M$  with  $c(0) = p, c'(t) \notin \mathcal{N}_{-1}|_{\gamma(t)}$ . Hence  $D = 0$ , as  $m/2 \geq 2$ . This,

together with the vanishing of the Christoffel symbols  $\Gamma_{ij}^k$  for  $i, j, k = 1, \dots, m$ , means the Lie group is  $G = \mathbb{R} \ltimes_{e^C} \mathbb{R}^m$ , where  $C = \begin{pmatrix} I_{m/2} & 0 \\ 0 & -I_{m/2} \end{pmatrix}$ .

Since  $M$  is complete, its universal covering is isometric to  $G$ . Finally, we show that  $G$  indeed admits quotients of finite volume, using Filipkiewicz's criterion (Lemma 4.2). In fact

$$\exp\left(\left(\log \frac{3+\sqrt{5}}{2}\right)C\right) = \frac{1}{2} \begin{pmatrix} (3+\sqrt{5})I_{m/2} & 0 \\ 0 & (3-\sqrt{5})I_{m/2} \end{pmatrix}$$

is conjugate to

$$\begin{pmatrix} 3 & -1 & & \\ \ddots & \ddots & \ddots & \\ & 3 & -1 & \\ 1 & 0 & & \\ \ddots & \ddots & & \\ 1 & 0 & & \end{pmatrix}.$$

This finishes the proof of Theorem 1.5.

## 6 Bracket-generation of the $\kappa$ -conullity distribution

Let  $M$  be a connected complete Riemannian manifold with nontrivial  $\kappa$ -nullity distribution  $\mathcal{N}_\kappa$  of constant rank, for some fixed  $\kappa \geq 0$ . In this section, we revisit some results related to the following question: When can two points in  $M$  be joined by a piecewise smooth curve always orthogonal to the  $\kappa$ -nullity distribution? An answer is given by Theorem 1.6, which we now prove.

### 6.1 The case $\kappa > 0$

This is part (a) of the theorem and the answer is easy. By Corollary 2.2,  $\mathcal{D} = \mathcal{N}_\kappa^\perp$  is bracket generating of step 2, and hence, owing to the Chow-Rashevskii theorem [27, Thm. 2.1.2], any two points in  $M$  can be joined by a piecewise smooth curve which is tangent to  $\mathcal{D}$  at smooth points.

### 6.2 The case $\kappa = 0$

Next we deal with part (b). We put  $\mathcal{D} = \mathcal{N}_0^\perp$  and recall the distributions  $\mathcal{D}^r$  introduced in (2.7). We also set  $\mathcal{E}^r = (\mathcal{D}^r)^\perp$  for  $r \geq 1$ .

**Lemma 6.1** *Let  $T \in \mathcal{E}^2$ ,  $X \in \mathcal{D}$  and  $Y \in TM$ . Then  $\langle \nabla_Y T, X \rangle = 0$ .*

**Proof** Since  $T \perp \mathcal{D}^2$ , the calculation (2.2) says that  $C_T$  is a symmetric endomorphism of  $\mathcal{D}$  and hence all of its eigenvalues are real. Now Corollary 2.1 implies that  $C_T \equiv 0$ . Finally, we decompose  $Y = Y^h + Y^v$  according to  $TM = \mathcal{D} \oplus \mathcal{N}_0$  and recall that  $\mathcal{N}_0$  is totally geodesic to obtain

$$\langle \nabla_Y T, X \rangle = -\langle C_T Y^h, X \rangle + \langle \nabla_{Y^v} T, X \rangle = 0,$$

as wished.  $\square$

**Lemma 6.2** *Let  $T \in \mathcal{E}^{r+1}$ ,  $Y \in \mathcal{D}^r$  and  $X \in \mathcal{D}$ , for some  $r \geq 1$ . Then  $\langle \nabla_X T, Y \rangle = 0$ .*

**Proof** We compute

$$\begin{aligned} \langle \nabla_X T, Y \rangle &= -\langle T, \nabla_X Y \rangle \\ &= -\langle T, \nabla_X Y \rangle - \langle \nabla_Y T, X \rangle \quad (\text{by Lemma 6.1}) \\ &= \langle T, [Y, X] \rangle \\ &= 0 \quad (\text{since } [Y, X] \in \mathcal{D}^{r+1}), \end{aligned}$$

as desired.  $\square$

For a point  $p \in M$ , consider the integers  $n_i(p) = \dim \mathcal{D}^i|_p$  and note that the non-decreasing sequence  $n_1(p), n_2(p), \dots$  obviously stabilizes, say at  $r = r(p)$ . The *growth vector* of  $\mathcal{D}$  at  $p \in M$  is the integer list  $(n_1(p), \dots, n_r(p))$ . The distribution  $\mathcal{D}$  is called *regular* at  $p$  if the growth vector is locally constant at  $p$ . The subset  $M_{reg}$  of regular points for  $\mathcal{D}$  is open and dense, and consists precisely of the points of  $M$  where all the  $\mathcal{D}^i$ 's are genuine distributions. Since the growth vector is a lower semicontinuous function,  $M_{reg}$  in particular contains the open set where the growth vector is maximal (with respect to the lexicographic order starting at  $n_1$ ).

Let  $\gamma$  be a nullity geodesic. Since  $\mathcal{D}$  is parallel along  $\gamma$  and the parallel transport along  $\gamma$  is an isometry, we deduce that the growth vector is constant along  $\gamma$ . It follows that the connected components of  $M_{reg}$  are foliated by the leaves of  $\mathcal{N}_0$ . Fix such a component, say  $U$ , with growth vector  $(n_1, \dots, n_r)$ .

Note that  $\mathcal{E}^{n_r} \neq 0$  if and only if  $\mathcal{D}$  is not bracket-generating on  $U$ . By the argument above,  $\mathcal{E}^{n_r}$  is parallel along a nullity geodesic. Moreover  $\mathcal{E}^{n_r+1} = \mathcal{E}^{n_r}$  so, owing to Lemma 6.2,  $\nabla_X T \in \mathcal{E}^{n_r}$  for all  $X \in \mathcal{D}$  and  $T \in \mathcal{E}^{n_r}$ . We have shown that  $\mathcal{E}^{n_r}$  is a parallel distribution in  $M$ . Note that the leaves of  $\mathcal{E}^{n_r}$  are isometric to a flat Euclidean space  $\mathbb{R}^s$  for some  $s \geq 0$  ( $s = 0$  corresponds to the case in which  $\mathcal{D}$  is bracket-generating in  $U$ ). By the de Rham decomposition theorem and the Chow-Rashevskii theorem [27, Thm. 2.1.2], and shrinking  $U$  if necessary,  $U$  splits as a Riemannian product  $U_0 \times \mathbb{R}^s$ , where  $s \geq 0$  (compare [23, Prop. 5.2 of Ch. IV]), and any two points of  $U_0 \times \{0\}$  can be joined by a piecewise smooth curve in  $U_0 \times \{0\}$  tangent to  $\mathcal{D}|_{U_0 \times \{0\}}$  at smooth points.

If  $M$  is homogeneous and simply-connected then  $U = M$ . The irreducibility of  $M$  implies  $s = 0$ . This proves (b) and finishes the proof of the theorem.

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